P.G. Semester-II Examination, 2023

MATHEMATICS

Course ID : 22152 **Course Code : MATH202C**

Course Title : Topology

Time : 2 Hours Full Marks : 40

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meaning.

Answer any **five** of the following questions: 8×5=40

1. a) Construct all possible topologies on $X = \{1,2,3\}$. 3

b) Prove that a topological space X is normal if and only if for each closed set E and each open set U on X with $E \subset U$, there exists an open set V such that $E \subset V \subset Cl(V) \subset U$. 4

c) For a subset A of a topological space X, prove that $Ext(Ext(A)) \supset Ext(X - A)$. 1

2. a) Let $X = (a, b, c\}$ and $Y = \{1, 2, 3, 4\}$. Construct a base with at least three sets of a topology on X and a base with at least four sets of a topology on Y. Hence construct a base of the product topology on $X \times Y$. 3

b) Let A and B be two subsets of a topological space X. Is $Cl(A \cap B) = Cl(A) \cap Cl(B)$ true? Justify your answer. 2

c) Prove that the union of two nowhere dense sets in a topological space X is nowhere dense in X. 3

3. a) Suppose X and Y are two topological spaces, and $f : X \to Y$ is a continuous function. Prove that the set $A = \{(x, y) \mid x, y \in X,\ f(x) = f(y)\}$ is closed in the topological space $X \times Y$ if Y is Hausdorff. 3

b) Suppose that for every pair of nonempty closed subsets E, F of a topological space X with $E \cap F = \emptyset$ there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for all $x \in E$, $f(x) = 1$ for all $x \in F$. Prove that X is normal. 3

c) Let

$\tau = \{\varnothing, X, \{a\}, \{a, b\}\}$ and $\sigma = \{\varnothing, Y, \{1\}, \{2\}, \{1, 2\}\}$ be topologies on $X = \{a, b, c\}$ and $Y = \{1, 2, 3\}$ respectively. A function $f : X \to Y$ is defined by $f(a) = 2$, $f(b) = 1$, $f(c) = 3$. Verify whether the function f continuous and open. 2

4. a) Suppose that G and H are open sets in the topological space X with $X = G \cup H$. Also suppose that there exist two functions $f_G : G \to Y$ and $f_H : H \to Y$ such that $f_G(x) = f_H(x)$ for all $x \in G \cap H$. Prove that the function $f : X \to Y$ defined by $f(x) = f_G(x),\ x \in G$ and $f(x) = f_H(x),\ x \in H$ is continuous on X if f_G and f_H both are continuous on G and H respectively. 3

b) Prove that each collection of closed sets in a compact topological space X with finite intersection property has a nonempty intersection. 4

c) State Tietze's extension theorem. 1

5. a) Prove that a function $f : X \to Y$ is open if and only if $f(Int_X(A)) \subset Int_Y(f(A))$ where $A \subset X$. 4

b) Prove that a compact Hausdorff space is normal. 4

6. a) If A and B are two disjoint compact subsets of a Hausdorff space X, prove that there exist open sets U, V in X such that $A \subset U$, $B \subset V$ and $U \cap V = \varnothing$. 4

b) Give an example with justification of a topological space X such that X is connected but each subspace $X - \{x\},\ x \in X$ of X is disconnected. 3

c) Is the derived set of any subset of $\mathbb{R}$ endowed with usual topology closed? Justify. 1

7. a) Prove that a topological space X is a T_1-space if and only if each $A \subset X$ is the intersection of all open sets containing A. Is A open in X? Justify your answer. 3+1

b) Show that a metrizable topological space is regular. 3

c) Show that there exist at least two clopen sets other than Ø and X in a disconnected topological space. 1

8. a) Let B be a collection of open sets of a topological space X and U be open in X. Suppose for each $x \in U$ there exists a $G \in B$ such that $x \in U \subset G$. Prove that there exists a subcollection C of B such that $\cup_{H \in C} H = U$. 2

b) Give an example with justification of a topological space to show that a regular space need not be Hausdorff. 2

c) If A and B are two mutually separated sets in a topological space X and E is connected in the subspace $Y = A \cup B$, prove that either $E \subset A$ or $E \subset B$. 4

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